



TITLE:

On weakly Φ -like of order α with respect to certain analytic functions (Conditions for Univalence of Functions and Applications)

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On weakly Φ -like of order α with respect to certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk E , weakly Φ -like of order α with respect to a function $g(z)$ is introduced. The purpose of the present paper is to drive univalence for weakly Φ -like of order α with respect to $g(z)$.

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1. Introduction

Let A be the class of functions of form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is called starlike if $f(z)$ satisfies the condition

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in E).$$

For $f(z)$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E)) \in \mathbb{C}^2$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$, and for arbitrary $\omega \in f(E)$, $\Phi(\omega, g(re^{i\theta}))$ ($0 < r < 1$ and $0 \leq \theta \leq 2\pi$) satisfy

$$\frac{d}{d\theta} \arg \Phi(\omega, g(re^{i\theta})) < \frac{1}{2}(3 - \alpha) \quad (z \in E)$$

where $1 < \alpha < 2$.

A function is said to be weakly Φ -like of order α with respect to a function $g(z)$ which satisfies the above condition of upper order $\frac{1}{2}(3 - \alpha)$ if it satisfies

$$(1.3) \quad \left| \arg \frac{zf'(z)}{\Phi(f(z), g(z))} \right| < \frac{\pi}{2} \alpha \quad (z \in E)$$

for $1 < \alpha < 2$ (cf. [1] and [2]).

2. Main result

Theorem 1. *If $f(z)$ is a weakly Φ -like of order α with respect to a function $g(z)$ of upper order $\frac{1}{2}(3 - \alpha)$ for $1 < \alpha < 2$, then $f(z)$ is univalent in E .*

Proof. We will prove it by reductive absurdity. Let us suppose that there exists a positive real number r ($0 < r < 1$) for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$.

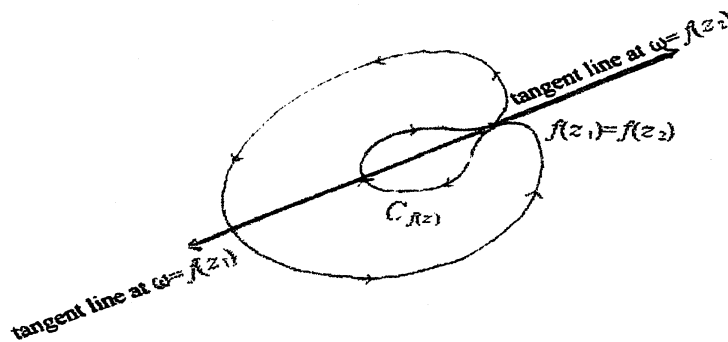


Figure 1

In view of Figure 1, we know that there are two points z_1 and z_2 ($z_1 \neq z_2$) such that

$$|z_1| = |z_2| = r, \quad z_1 = re^{i\theta_1}, \quad z_2 = re^{i\theta_2}, \quad 0 < \beta = \theta_2 - \theta_1,$$

for which $f(z_1) = f(z_2)$. Let us put $C = \{z : z = re^{i\theta}, \theta_1 < \theta \leq \theta_2\}$ and $C_{f(z)} = \{f(z) : z \in C\}$.

On the other hand, from the assumption of the theorem, we have

$$\begin{aligned} & \int_{|z|=r} d \arg d \frac{zf'(z)}{\phi(f(z), g(z))} \\ &= \int_{|z|=r} d \arg z + \int_{|z|=r} d \arg df(z) - \int_{|z|=r} d \arg dz - \int_{|z|=r} d \arg \phi(f(z), g(z)) \\ &= 2\pi + \int_{|z|=r} d \arg df(z) - 2\pi - \int_{|z|=r} d \arg \phi(f(z), g(z)) \end{aligned}$$

and

$$\pi\alpha > \int_{|z|=r} d \arg df(z) - \int_{|z|=r} d \arg \phi(f(z), g(z)) > -\pi\alpha.$$

Now then, it is trivial that

$$\int_{|z|=r} d \arg \phi(f(z), g(z)) = 2\pi.$$

This shows that

$$4\pi > \pi\alpha + 2\pi > \int_{|z|=r} d \arg df(z) > 2\pi - \pi\alpha > 0$$

and therefore it must be

$$(2.1) \quad \int_{|z|=r} d \arg df(z) = 2\pi.$$

Now, we have

$$(2.2) \quad \int_{C_{f(z)}} d \arg df(z) = \int_C d \arg f'(z) dz = -\pi.$$

Putting $L = \{z : |z| = r\}$, then from the assumption of the theorem, we have

$$\pi\alpha > \int_{L-C} d \arg \frac{zf'(z)}{\Phi(f(z), g(z))} > -\pi\alpha$$

and so, we have

$$\pi\alpha > \int_{L-C} d \arg df(z) - \int_{L-C} d \arg \Phi(f(z), g(z)) > -\pi\alpha.$$

It follows that

$$\begin{aligned}
 & \int_{L-C} d \arg df(z) \\
 & < \pi\alpha + \arg \Phi(f(z_1), g(re^{i(\theta_1+2\pi)})) - \arg \Phi(f(z_2), g(re^{i\theta_2})) \\
 & = \pi\alpha + \arg \Phi(f(z_2), g(re^{i(\theta_1+2\pi)})) - \arg \Phi(f(z_2), g(re^{i\theta_2})) \\
 & = \pi\alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{d}{d\theta} \arg \Phi(f(z_2), g(re^{i\theta})) d\theta \\
 & < \pi\alpha + \int_{\theta_2}^{\theta_1+2\pi} \frac{1}{2}(3-\alpha) d\theta \\
 & = \pi\alpha + \frac{1}{2}(3-\alpha)(2\pi - \beta) \\
 & < \pi\alpha + (3-\alpha)\pi = 3\pi
 \end{aligned}$$

This show that

$$(2.3) \quad \int_{|z|=r} d \arg df(z) - \int_{C_{f(z)}} d \arg df(z) < 3\pi.$$

From (2.1), (2.2) and (2.3), we have contradiction. This completes the proof of theroem.

Remak. When $f(z)$ satisfies the hypothesis of Theorem 1, the real part of the function $zf'/\Phi(f(z), g(z))$ can be negative.

Theorem 2. Let $\Phi(f(z), g(z))$ be analytic on $(f(E), g(E))$ with $\Phi(f(0), g(0)) = 0$, $\Phi(f(z), g(z)) \neq 0$ and $f(z) \neq 0$ in $0 < |z| < 1$ and for arbitrary $w \in f(E)$, $\Phi(w, g(re^{i\theta}))$ satisfies the following condition

$$\frac{d}{d\theta} \arg \Phi(w, g(re^{i\theta})) > -\frac{1}{2} \quad (z \in E)$$

where $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Then, if $f(z)$ satisfies the following conditon

$$\operatorname{Re} \frac{z^2(f'(z))^2}{\Phi(f(z), g(z))} > 0 \quad (z \in E)$$

then $f(z)$ is univalent in E .

Proof. Applying the same method as the proof of Theorem 1, let us suppose that there exists a positive real number r ($0 < r < 1$) for which $f(z)$ is univalent in $|z| < r$, but $f(z)$ is not univalent on $|z| = r$. Also, in view of Figure 1, we know that there are two points z_1 and z_2 ($z_1 \neq z_2$) such that $|z_1| = |z_2| = r$, $z_1 = re^{i\theta_1}$, $z_2 = re^{i\theta_2}$, $0 < \theta_2 - \theta_1$, for which $f(z_1) = f(z_2)$. From the assumption of the theorem, we have

$$\begin{aligned}
\pi &> \int_C d \arg \frac{z^2 (f'(z))^2}{\Phi(f(z), g(z))} \\
&= \int_C d \arg z^2 + \int_C d \arg (f'(z))^2 - \int_C d \arg \Phi(f(z), g(z)) \\
&= 2 \int_C d \arg z + 2 \int_C d \arg f'(z) - \int_C d \arg \Phi(f(z), g(z)) \\
&= 2 \int_C d \arg z + 2 \int_C d \arg df(z) - 2 \int_C d \arg dz - \int_C d \arg \Phi(f(z), g(z)) \\
&= 2 \int_C d \arg df(z) - (\arg \Phi(f(z_2), g(re^{i\theta_2})) - \arg \Phi(f(z_1), g(re^{i\theta_1}))) \\
&= 2 \int_C d \arg df(z) - (\arg \Phi(f(z_1), g(re^{i\theta_2})) - \arg \Phi(f(z_1), g(re^{i\theta_1}))) \\
&> -\pi.
\end{aligned}$$

It follows that

$$\begin{aligned}
&2 \int_C d \arg df(z) \\
&> (\arg \Phi(f(z_1), g(re^{i\theta_2})) - \arg \Phi(f(z_1), g(re^{i\theta_1}))) - \pi \\
&= \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} \arg \Phi(f(z_1), g(re^{i\theta})) d\theta - \pi \\
&> -\frac{1}{2} \int_{\theta_1}^{\theta_2} d\theta - \pi \\
&> -\pi - \pi = -2\pi
\end{aligned}$$

and therefore, we have

$$\int_C d \arg df(z) > -\pi,$$

but from the assumption, we have

$$\int_C d \arg df(z) = -\pi.$$

This is a contradiction and it completes the proof.

Applying the proof of Theorem 2, we have following corollary:

Corollary 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in E and suppose that

$$\operatorname{Re} \frac{z^2 (f'(z))^2}{f(z)^{2-\beta} g(z)^\beta} > 0 \quad (z \in E)$$

where $\beta > 0$ and

$$(2.4) \quad \operatorname{Re} \frac{zg'(z)}{g(z)} > -\frac{1}{2\beta} \quad \text{in } E,$$

then $f(z)$ is univalent in E .

Remark. If $g(z)$ satisfies the condition (2.4), then $g(z)$ is not necessarily a starlike function.

References

- [1] L. Brickman, *Φ -like analytic functions*, Bull. Amer. Math. Soc. **79** (1973), 555–558.
- [2] M. Nunokawa, S. Owa, O. S. Kwon and N. E. Cho, *On Φ -like with respect to certain starlike functions*, to appear.